Applied Mathematics Letters 26 (2013) 160–164

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A new fourth-order numerical scheme for option pricing under the CEV model

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ARTICLE INFO

Article history:
Received 4 May 2012
Received in revised form 3 August 2012
Accepted 3 August 2012

Keywords:
Option pricing
Constant elasticity of variance model
High-order compact schemes
Grid refinement

ABSTRACT

The empirically observed negative relationship between a stock price and its return volatility can be captured by the constant elasticity of variance option pricing model. For European options, closed form expressions involve the non-central chi-square distribution whose computation can be slow when the elasticity factor is close to one, volatility is low or time to maturity is small. We present a fast numerical scheme based on a high-order compact discretisation which accurately computes the option price. Various numerical examples indicate that for comparable computational times, the option price computed with the scheme has higher accuracy than the Crank–Nicolson numerical solution. The scheme accurately computes the hedging parameters and is stable for strongly negative values of the elasticity factor.

1. Introduction

Empirical evidence and theoretical arguments (see [1] and references therein) support the hypothesis that there is an association between stock price and volatility. To account for this, Cox [2] introduced the constant elasticity of variance (CEV) model which nests the constant volatility lognormal diffusion process of Black–Scholes. Empirical research has shown that the CEV model provides a better fit to observed market option prices.

For this model, under the risk-neutral measure the stock price process S is assumed to follow the dynamics

$$dS = rSdt + \sigma S^{\alpha}\, dW, \tag{1}$$

where α is the elasticity factor, r is the risk-free interest rate and W is standard Brownian motion. The model (1) with negative α exhibits implied volatility smiles similar to the smiles observed in index options and Jackwerth and Rubinstein [3] empirically observed that the elasticity factor α can be as low as -3 in the index options market.

Evaluating the closed-form expression [1] for European option prices under the CEV model requires the computation of the noncentral chi-square distribution. For elasticity factors close to one, computation of the option price via the analytical formula is computationally expensive. Much research has focused on efficient computations of the non-central chi-square distribution. Numerically solving the pricing equation is a good alternative but there exists few numerical techniques. Wong and Zhao [4] have proposed a Crank–Nicolson scheme for pricing European and American options.

This work proposes a more accurate scheme for the CEV European option pricing problem. Our technique is based on a high-order discretisation and thus has the advantage of computing sufficiently accurate option prices using relatively coarse mesh sizes. We provide numerical evidence that for approximately the same computational time, our scheme is more accurate than the Crank–Nicolson scheme and in addition, uniform fourth-order convergence behaviour is achieved for $\alpha \geq -3$.

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doi:10.1016/j.aml.2012.08.002

Our scheme is based on a Crandall discretisation [5] of the pricing equation and to achieve the high-order convergence rate, we refine the grid near the strike price as in [6] because of the non-smooth payoff function. The resulting technique is fast, exhibits a fourth-order convergence rate and is successful in computing highly accurate option prices even for strongly large negative values of the elasticity factor. For such difficult test problems, the matrices arising in the solution step of some high-order discretisation techniques may become nearly singular. No such problems are observed with our scheme.

The structure of this paper is as follows. In Section 2, we present the discretisation of the pricing equation and in Section 3, we describe numerical results illustrating the good properties of the numerical scheme. Our conclusions are presented in Section 4.

2. CEV pricing equation and numerical discretisation

Under the CEV model, a European put with strike K has price $V(S, t)$ which is the solution of the initial–boundary value problem

$$V_t = \frac{1}{2}\sigma^2 S^{2\alpha} V_{SS} + rSV_S - rV, \quad S \geq 0,\ 0 \leq t \leq T, \tag{2}$$

with initial condition $V(S, 0) = \max(K - S, 0)$ and boundary conditions given by $V(0, t) = Ke^{-rt}$ and $V(S, t) \to 0$ as $S \to +\infty$.

If we let $f(S, V, V_t, V_S) = (V_t - rSV_S + rV)/a(S)$, where $a(S) = \sigma^2 S^{2\alpha}/2$, then the pricing equation (2) can be expressed in the form

$$V_{SS} = f(S, V, V_t, V_S), \quad S > 0,\ 0 \leq t \leq T. \tag{3}$$

2.1. Fully discrete high-order compact (HOC) scheme

The numerical scheme is based on localising problem (3) to a finite domain $\Omega = [S_{\min}, S_{\max}] \times [0, T]$ and then shifting the boundary conditions to the right and left boundaries of this domain. Consider a uniform grid Ω_h^k with a mesh spacing of $h = (S_{\max} - S_{\min})/M$ in the S-direction and a spacing of $k = T/N$ in the time direction. Let $a_m = a(S_m)$ and denote the option price $V(S_m, t_n) = V_m^n$ where $S_m = S_{\min} + mh$ for $m = 0, 1, \ldots, M$ and $t_n = nk$ for $n = 0, 1, \ldots, N$. A Numerov discretisation [5] of (3) on Ω_h^k has the form

$$\frac{1}{h^2}\delta_S^2 V_m^{n+\frac{1}{2}} = \frac{1}{12}\left(f_{m+1}^{n+\frac{1}{2}} + 10f_m^{n+\frac{1}{2}} + f_{m-1}^{n+\frac{1}{2}}\right), \tag{4}$$

where

$$\delta_S^2 V_m^{n+\frac{1}{2}} \approx \left(V_{m-1}^{n+\frac{1}{2}} - 2V_m^{n+\frac{1}{2}} + V_{m+1}^{n+\frac{1}{2}}\right), \qquad V_m^{n+\frac{1}{2}} \approx \frac{1}{2}\left(V_m^{n+1} + V_m^n\right), \tag{5}$$

$$f_{m+q}^{n+\frac{1}{2}} \approx f\left(S_{m+q},\ V_{m+q}^{n+\frac{1}{2}},\ \frac{V_{m+q}^{n+1} - V_{m+q}^n}{k},\ (V_S)_{m+q}^{n+\frac{1}{2}}\right), \quad q = -1, 0, 1.$$

For $(V_S)_{m\pm1}^{n+\frac{1}{2}}$ we use standard second-order approximations given by

$$(V_S)_{m\pm1}^{n+\frac{1}{2}} \approx \pm\frac{1}{2h}\left(3V_{m\pm1}^{n+\frac{1}{2}} - 4V_m^{n+\frac{1}{2}} + V_{m\mp1}^{n+\frac{1}{2}}\right).$$

However a correction term in the approximation for $(V_S)_m^{n+\frac{1}{2}}$ of the form

$$(V_S)_m^{n+\frac{1}{2}} \approx \frac{1}{2h}\left(V_{m+1}^{n+\frac{1}{2}} - V_{m-1}^{n+\frac{1}{2}}\right) + \lambda h\left(f_{m+1}^{n+\frac{1}{2}} - f_{m-1}^{n+\frac{1}{2}}\right) \tag{6}$$

is required in order to obtain a scheme having fourth-order accuracy. This can be seen from the expression for the resulting truncation error

$$T_m^n = \frac{h^2}{12}(1 + 20\lambda)V_{(3,0)}F_{(0,0)}^{(1,0)} + \frac{k^2}{24}T_1 + \frac{h^2 k}{72}T_2 + \frac{h^4}{2160}T_3 + \mathcal{O}\left(k^3 + h^2k^2\right), \tag{7}$$

where T_1, T_2 and T_3 are written in terms of [5]

$$V^{(i,j)} = \frac{\partial^{i+j}V}{\partial S^i \partial t^j}, \quad \text{and} \quad F_{(p,q)}^{(i,j)} = \frac{\partial^{i+j+p+q}f}{\partial t^i \partial V^j \partial V_S^p \partial V_t^q},$$

with the expressions for $V^{(i,j)}$ and $F_{(p,q)}^{(i,j)}$ evaluated at the grid point (S_m, t_n).

From (7) we find that a fourth-order scheme is possible only when the parameter $\lambda = -1/20$ in (6) and the time step k is chosen as $k = \mu h^2$.

Table 1
European put option prices for $K = 110$ and $\alpha = 0$ and hedging parameters. The exact option price is 9.955171.

M	CN				HOC			
	Price	Error	Order	cpu (s)	Price	Error	Order	cpu (s)
2^5	9.850392	1.0(−01)	–	0.003	9.920227	3.5(−02)	–	0.005
2^6	9.927739	2.7(−02)	1.933	0.005	9.953757	1.4(−03)	4.627	0.008
2^7	9.948297	6.9(−03)	1.997	0.009	9.955087	8.4(−05)	4.074	0.014
2^8	9.953451	1.7(−03)	1.999	0.024	9.955166	4.8(−06)	4.113	0.047
2^9	9.954741	4.3(−04)	1.999	0.050	9.955171	2.8(−07)	4.091	0.222
Greeks	$\Delta = -0.699007$		$\Gamma = 0.024931$		$\Delta = -0.698981$		$\Gamma = 0.024931$	

Table 2
European put option prices for $K = 110$ and $\alpha = 2/3$ and hedging parameters. The exact option price is 10.109899.

M	CN				HOC			
	Price	Error	Order	cpu (s)	Price	Error	Order	cpu (s)
2^5	10.006977	1.0(−01)	–	0.009	10.075263	3.5(−02)	–	0.003
2^6	10.082675	2.7(−02)	1.919	0.014	10.108630	1.3(−03)	4.771	0.009
2^7	10.103080	6.8(−03)	1.997	0.025	10.109824	7.5(−05)	4.089	0.017
2^8	10.108193	1.7(−03)	1.999	0.055	10.109894	4.2(−06)	4.155	0.044
2^9	10.109472	4.3(−04)	1.999	0.125	10.109898	2.4(−07)	4.132	0.222
Greeks	$\Delta = -0.676799$		$\Gamma = 0.025471$		$\Delta = -0.676777$		$\Gamma = 0.025470$	

Substituting the approximations (5)–(6) in (4) gives the fully discrete scheme

$$b_{m-1}V_{m-1}^{n+1} + b_m V_m^{n+1} + b_{m+1}V_{m+1}^{n+1} = c_{m-1}V_{m-1}^{n} + c_m V_m^{n} + c_{m+1}V_{m+1}^{n}, \tag{8}$$

where

$$\begin{aligned} b_{m\pm1} &= 2a_m\left(a_{m\mp1}\left[-24ka_{m\pm1} + h(4h + 2hkr \mp 3krS_{m\pm1})\right] \pm hkra_{m\pm1}S_{m\mp1}\right) \\ &\quad \pm hrS_m\left(a_{m\mp1}[-20ka_{m\pm1} + h(4h + 2khr \mp 3krS_{m\pm1})] \mp hkra_{m\pm1}S_{m\mp1}\right), \\ c_{m\pm1} &= 2a_m\left(a_{m\mp1}[24ka_{m\pm1} + h(4h - 2hkr \pm 3krS_{m\pm1})] \mp hkra_{m\pm1}S_{m\mp1}\right) \\ &\quad \pm hrS_m\left(a_{m\mp1}[20ka_{m\pm1} + h(4h - 2khr \pm 3krS_{m\pm1})] \pm hkra_{m\pm1}S_{m\mp1}\right), \\ b_m &= 4\left(a_{m-1}[2a_{m+1}(5h^2(kr+2) + 12ka_m) + khr(hrS_m + 2a_m)S_{m+1}] + hkra_{m+1}S_{m-1}[hrS_m - 2a_m]\right), \\ c_m &= -4\left(a_{m-1}[2a_{m+1}(5h^2(kr-2) + 12ka_m) - khr(hrS_m + 2a_m)S_{m+1}] + hkra_{m+1}S_{m-1}[hrS_m - 2a_m]\right). \end{aligned}$$

The option pricing problem has a non-smooth payoff function $V(S, 0)$ and in order to obtain numerical solutions which exhibit fourth-order convergence ratios, we employ a grid refinement technique near the strike K. This technique was employed by Zhang and Sun for multigrid solution of convection–diffusion problems [6].

3. Numerical results

Numerical experiments have been performed using Mathematica 7 on a Core i7 laptop with 8 GB RAM and speed 3.20 GHz. Implementation details are as follows. Computed option prices are given for positive and negative values of the elasticity factor α and the merit of our scheme is illustrated in its ability to accurately compute the price for large negative α.

In all our examples, we price a European put option with a current stock price of $S = 100$, a maturity of $T = 0.5$ year, an interest rate of $r = 0.05$ and an at-the-money volatility of $\sigma = 20\%$. We choose $S_{\min} = 1$ and $S_{\max} = 2K - S_{\min}$ to ensure that the strike price K belongs to the uniform mesh. For the Crank–Nicolson scheme, we use the same number of spatial and temporal steps, that is, $M = N$. For the new scheme, the choice of the parabolic mesh ratio $\mu = k/h^2$ is motivated by our aim to obtain uniform fourth-order convergence rates and to obtain numerical solutions having six-figure accuracy on a mesh with $M = 512$. The truncation error given in (7) indicates that the time step should be sufficiently small to obtain high accuracy. We experimented with different values of the parabolic mesh ratio and we observed that the choice $\mu = 3/1000$ gave good numerical results.

Table 1 shows numerical results for the case $\alpha = 0$. Computed option prices, errors (difference between exact and computed prices), convergence rates, cpu timings and the Greeks Delta (Δ) and Gamma (Γ) are shown for our high-order compact (HOC) scheme and the Crank–Nicolson (CN) discretisation which is based on second-order approximation of the spatial derivatives. We observe that both schemes achieve the expected convergence rates (order 2 for CN and order 4 for HOC). More significantly, HOC yields a more accurate numerical solution for comparable computational times.

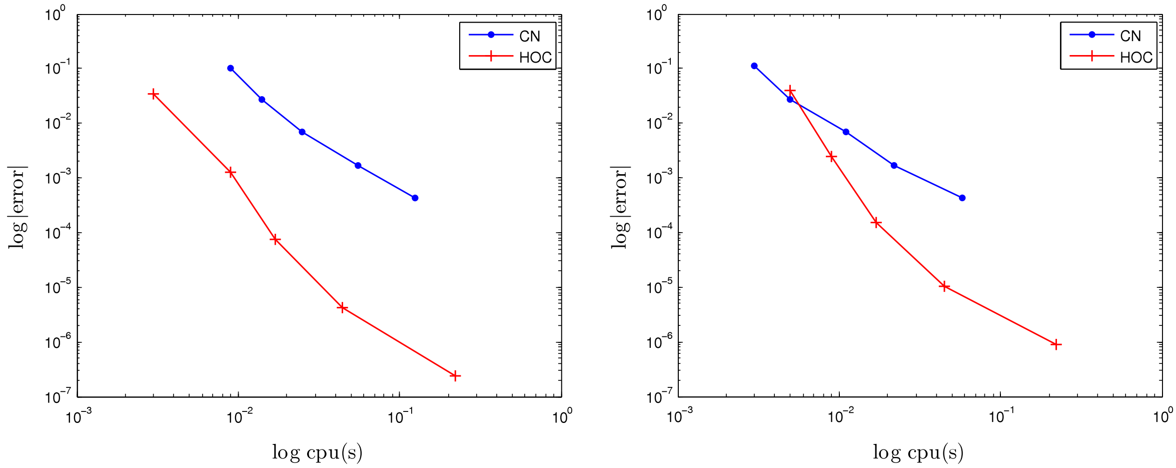
Results for two cases with $\alpha = 2/3$ and $\alpha = -3$ are given in Tables 2–3 respectively. The results for both cases indicate that HOC computes the option price with high accuracy. Similar results are obtained for other values of the strike price K.

Table 3
European put option prices for $K = 110$ and $\alpha = -3$ and hedging parameters. The exact option price is 9.348571.

M	CN				HOC			
	Price	Error	Order	cpu (s)	Price	Error	Order	cpu (s)
2^5	9.237679	1.1(−01)	–	0.003	9.308772	4.0(−02)	–	0.005
2^6	9.321298	2.7(−02)	2.024	0.005	9.346153	2.4(−03)	4.041	0.009
2^7	9.341723	6.8(−03)	1.994	0.011	9.348416	1.5(−04)	3.968	0.017
2^8	9.346858	1.7(−03)	1.999	0.022	9.348561	1.0(−05)	3.917	0.045
2^9	9.348142	4.3(−04)	1.999	0.058	9.348570	9.0(−07)	3.510	0.221
Greeks	$\Delta = -0.783935$		$\Gamma = 0.022897$		$\Delta = -0.783895$		$\Gamma = 0.022901$	

Table 4
European put option prices and hedging parameters for negative values of α.

	M	$\alpha = -4$		$\alpha = -5$		$\alpha = -6$	
		$K = 90$	$K = 110$	$K = 100$	$K = 110$	$K = 90$	$K = 110$
CN	2^7	2.562980	9.172631	4.639230	9.010244	3.249786	8.857853
	2^8	2.565639	9.174367	4.640663	9.011567	3.252815	8.858697
	2^9	2.565905	9.174175	4.640840	9.011194	3.253304	8.858120
HOC	2^7	2.566251	9.173639	4.634967	9.010514	3.253671	8.857273
	2^8	2.566262	9.173818	4.642473	9.010739	3.253653	8.857555
	2^9	2.566262	9.173830	4.641350	9.010754	3.253652	8.857574
Exact		2.566765	9.173834	4.641232	9.010769	3.255940	8.857605
Δ_{CN}		−0.381687	−0.807976	−0.592787	−0.830238	−0.502035	−0.850622
Δ_{HOC}		−0.381689	−0.807930	−0.592756	−0.830187	−0.502007	−0.850566
Γ_{CN}		0.033651	0.022334	0.035561	0.021791	0.038867	0.021202
Γ_{HOC}		0.033648	0.022340	0.035560	0.021798	0.038864	0.021211



(a) CN and HOC for $\alpha = 2/3$.

(b) CN and HOC for $\alpha = -3$.

Fig. 1. Error and computational time for CN and HOC.

Fig. 1 shows a log plot of the accuracy against cpu time for the cases $\alpha = 2/3$ and $\alpha = -3$. The superiority of HOC over CN is clearly seen in the sense that for comparable running times, the HOC solution has a higher accuracy that the CN solution. For illustration, when $\alpha = 2/3$, HOC computes a solution with an error of 7.5×10^{-5} in 17 ms (milliseconds) compared to an error of 6.8×10^{-3} in 25 ms for CN.

Table 4 gives the computed option prices for negative α and we see that HOC is successful in accurately computing the option price for $\alpha = -4, -5$ and -6. For such values of α, CN yields ill-conditioned matrices but no such problems are observed with HOC.

4. Conclusion

A new high-order numerical scheme for solving the option pricing problem under the CEV model is introduced. The algorithm is fast and its ability to compute accurate prices for strongly large negative values of the elasticity factor is demonstrated. We aim to extend this efficient scheme to price American and path-dependent options in a future work.

Acknowledgements

The research of N. Thakoor was supported by a Postgraduate Research Scholarship from the Tertiary Education Commission and the University of Mauritius.

References

[1] M. Schroder, Computing the constant elasticity of variance option pricing formula, J. Finance 44 (1989) 211–219.
[2] J. Cox, The constant elasticity of variance option pricing model, J. Portfolio Manage. 22 (1996) 16–17.
[3] J. Jackwerth, M. Rubinstein, Recovering stochastic processes from option prices, Working Paper, University of California, Berkeley, CA, 2001.
[4] H.Y. Wong, J. Zhao, An artificial boundary method for American option pricing under the CEV model, SIAM J. Numer. Anal. 46 (2008) 2183–2209.
[5] M.K. Jain, R.K. Jain, R.K. Mohanty, A fourth order difference method for the one dimensional general quasilinear parabolic differential equation, Numer. Methods Partial Differential Equations 6 (1990) 311–319.
[6] J. Zhang, H. Sun, High-order compact scheme with multigrid local mesh refinement procedure for convection–diffusion problems, Comput. Methods Appl. Mech. Eng. 191 (2002) 4861–4874.